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Analytical Result for Dimensionally Regularized Massive On-Shell Planar Double Box

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Abstract

The dimensionally regularized master planar double box Feynman diagram with four massive and three massless lines, powers of propagators equal to one, all four legs on the mass shell, i.e. with $p_i^2 = m^2$, $i = 1, 2, 3, 4$, is analytically evaluated for general values of m^2 and the Mandelstam variables s and t . An explicit result is expressed in terms of polylogarithms, up to the third order, depending on special combinations of m^2 , s and t .

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1. Systematical analytical evaluation of two-loop Feynman diagrams with four external lines within dimensional regularization [1] began two and a half years ago. In the pure massless case with all end-points on-shell, i.e. $p_i^2 = 0$, $i = 1, 2, 3, 4$, the problem of analytical evaluation of two-loop four-point diagrams in expansion in $\epsilon = (4 - d)/2$, where d is the space-time dimension, has been completely solved in [2, 3, 4, 5, 6]. Any such diagram can be expressed, in Laurent expansion in ϵ up to a finite part, through polylogarithms and generalized polylogarithms up to the fourth order, depending on the ratio of the Mandelstam variables s and t . The corresponding analytical algorithms have been successfully applied to the evaluation of two-loop virtual corrections to various scattering processes [7] in the zero-mass approximation.

If in addition to the Mandelstam variables s and t there is one more massive parameter, the four-point diagrams become much more complicated. However, in the case of one leg off-shell relevant to the process $e^+e^- \rightarrow 3\text{jets}$ (see, e.g., [8]), the problem of the evaluation has been solved: master integrals have been evaluated [9, 10], either by use of Mellin–Barnes representation or the method of differential equations [11], and a reduction procedure has been developed [10]. (See [12] where the present status of NNLO calculations of the process $e^+e^- \rightarrow 3\text{jets}$ is characterized.)

The purpose of this paper is to turn attention to on-shell four-point diagrams with a non-zero internal mass and analytically evaluate the scalar double box diagram shown in Fig. 1. The calculational experience, in particular obtained in the above

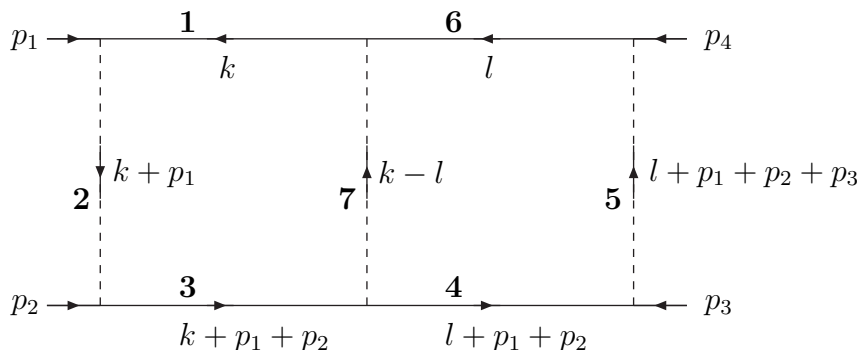


Figure 1: Planar double box diagram. Solid and dashed lines denote massive and massless propagators, respectively.

mentioned works on the evaluation of four-point diagrams, tell us that if such master integrals can be evaluated, the problem can be also completely solved, after evaluating other master integrals and constructing a recursive procedure that expresses any given Feynman integral with general numerators and integer powers of propagators through the master integrals. Therefore this explicit analytical result can be considered as a kind of existence theorem, in the sense that it strongly indicates the possibility to analytically compute various scattering processes in two loops without putting masses to zero.

To arrive at this result we derive in the next section an appropriate Mellin–Barnes (MB) representation for the general planar double box within dimensional regularization. Then we turn to the master double box and use a standard procedure of taking residues and shifting contours to resolve the structure of singularities in the parameter of dimensional regularization, ϵ . The initial MB integral is eventually decomposed into seven pieces where expansion of the integrand in ϵ becomes possible. After evaluating these expanded integrals we obtain an explicit analytical result expressed in terms of polylogarithms, up to the third order, depending on special combinations of m^2 , s and t . In the last section, leading order terms of the asymptotic expansion of the considered diagram in the limit of small m obtained by the strategy of expansion by regions [13, 14, 15] are presented. They serve as a crucial check of the result obtained. We conclude with a discussion of the results of the paper and open problems.

2. Let us consider the general on-shell double box diagram of Fig. 1, i.e. with general irreducible numerator and powers of propagators. We choose this irreducible numerator and the routing of the external momenta as in [6]. For convenience, we consider the factor with $(k + p_1 + p_2 + p_3)^2$ corresponding to the irreducible numerator as an extra propagator but, really, we are interested only in the non-positive integer values of a_8 . This general double box Feynman integral takes the form

$$B(a_1, \dots, a_8; s, t, m^2; \epsilon) = \int \int \frac{d^d k d^d l}{(k^2 - m^2)^{a_1} [(k + p_1)^2]^{a_2} [(k + p_1 + p_2)^2 - m^2]^{a_3} [(k + p_1 + p_2 + p_3)^2]^{-a_8} [(l + p_1 + p_2)^2 - m^2]^{a_4} [(l + p_1 + p_2 + p_3)^2]^{a_5} (l^2 - m^2)^{a_6} [(k - l)^2]^{a_7}}, \quad (1)$$

where $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$, and k and l are respectively loop momenta of the left and the right box. Usual prescriptions $k^2 = k^2 + i0$, $s = s + i0$, etc. are implied.

To resolve the singularity structure of Feynman integrals in ϵ it is very useful to apply the MB representation

$$\frac{1}{(X + Y)^\nu} = \frac{1}{\Gamma(\nu)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{Y^z}{X^{\nu+z}} \Gamma(\nu + z) \Gamma(-z), \quad (2)$$

that makes it possible to replace sums of terms raised to some power by their products in some powers, at the cost of introducing an extra integration. In [2, 4, 9] MB integrations were introduced directly in alpha/Feynman parametric integrals. It turns out more convenient to follow (as in [6]) the strategy of [16] and introduce, in a suitable way, MB integrations, first, after integration over one of the loop momenta, l , and complete this procedure after integration over the second loop momentum, k . In fact, the procedure of [6] is straightforwardly generalized by introducing two extra MB integrations when separating terms with m^2 after each of the integrations over the loop momenta, and after appropriate changes of variables leads to the following sixfold MB representation of (1):

$$B(a_1, \dots, a_8; s, t, m^2; \epsilon) = \frac{(i\pi^{d/2})^2 (-1)^a}{\prod_{j=2,4,5,6,7} \Gamma(a_j) \Gamma(4 - a_{4567} - 2\epsilon) (-s)^{a-4+2\epsilon}}$$

$$\begin{aligned}
& \times \frac{1}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} dw \prod_{j=1}^5 dz_j \left(\frac{m^2}{-s} \right)^{z_1+z_5} \left(\frac{t}{s} \right)^w \frac{\Gamma(a_2+w)\Gamma(-w)\Gamma(z_2+z_4)\Gamma(z_3+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)} \\
& \times \frac{\Gamma(4-a_{13}-2a_{28}-2\epsilon+z_2+z_3)\Gamma(a_{1238}-2+\epsilon+z_4+z_5)\Gamma(a_7+w-z_4)}{\Gamma(4-a_{46}-2a_{57}-2\epsilon-2w-2z_1-z_2-z_3)} \\
& \times \frac{\Gamma(a_{4567}-2+\epsilon+w+z_1-z_4)\Gamma(a_8-z_2-z_3-z_4)\Gamma(-w-z_2-z_3-z_4)}{\Gamma(4-a_{1238}-2\epsilon+w-z_4)\Gamma(a_8-w-z_2-z_3-z_4)} \\
& \times \frac{\Gamma(a_5+w+z_2+z_3+z_4)\Gamma(2-a_{567}-\epsilon-w-z_1-z_2)}{\Gamma(4-a_{13}-2a_{28}-2\epsilon+z_2+z_3-2z_5)} \\
& \times \Gamma(2-a_{457}-\epsilon-w-z_1-z_3)\Gamma(2-a_{128}-\epsilon+z_2-z_5)\Gamma(2-a_{238}-\epsilon+z_3-z_5) \\
& \times \Gamma(4-a_{46}-2a_{57}-2\epsilon-2w-z_2-z_3)\Gamma(-z_1)\Gamma(-z_5), \tag{3}
\end{aligned}$$

where $a_{4567} = a_4 + a_5 + a_6 + a_7$, $a_{13} = a_1 + a_3$, etc., and integration contours are chosen in the standard way.

In the case of the master double box, we set $a_i = 1$ for $i = 1, 2, \dots, 7$ and $a_8 = 0$ and obtain

$$\begin{aligned}
B^{(0)}(s, t, m^2; \epsilon) & \equiv B(1, \dots, 1, 0; s, t, m^2; \epsilon) \\
& = -\frac{(i\pi^{d/2})^2}{\Gamma(-2\epsilon)(-s)^{3+2\epsilon}} \frac{1}{(2\pi i)^6} \int_{-i\infty}^{+i\infty} dw \prod_{j=1}^5 dz_j \left(\frac{m^2}{-s} \right)^{z_1+z_5} \left(\frac{t}{s} \right)^w \frac{\Gamma(1+w)\Gamma(-w)}{\Gamma(1-2\epsilon+w-z_4)} \\
& \times \frac{\Gamma(2+\epsilon+w+z_1-z_4)\Gamma(-1-\epsilon-w-z_1-z_2)\Gamma(-1-\epsilon-w-z_1-z_3)\Gamma(-z_1)}{\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)\Gamma(-2\epsilon+z_2+z_3-2z_5)} \\
& \times \frac{\Gamma(-\epsilon+z_2-z_5)\Gamma(-\epsilon+z_3-z_5)\Gamma(1+\epsilon+z_4+z_5)\Gamma(-z_5)\Gamma(-2\epsilon+z_2+z_3)}{\Gamma(-2-2\epsilon-2w-2z_1-z_2-z_3)} \\
& \times \Gamma(-2-2\epsilon-2w-z_2-z_3)\Gamma(1+w+z_2+z_3+z_4)\Gamma(z_2+z_4)\Gamma(z_3+z_4) \\
& \times \Gamma(1+w-z_4)\Gamma(-z_2-z_3-z_4). \tag{4}
\end{aligned}$$

Observe that, because of the presence of the factor $\Gamma(-2\epsilon)$ in the denominator, we are forced to take some residue in order to arrive at a non-zero result at $\epsilon = 0$, so that the integral is effectively fivefold.

The resolution of singularities in ϵ is performed also in the standard way (see [2, 4, 6, 9]) and reduces to shifting contours and taking residues. The goal of this procedure is to decompose a given integral into pieces where the Laurent expansion in ϵ of the integrand becomes possible. This is how such procedure can be performed for (4):

1. Take minus residue at $z_3 = -2-2\epsilon-2w-z_2$, then minus residue at $w = -1-2\epsilon$, then residue at $z_4 = 0$, then residue at $z_2 = 0$, expand in a Laurent series in ϵ up to a finite part. Let us denote the resulting integral over z_1 and z_5 by B_1 .
2. Take minus residue at $z_3 = -2-2\epsilon-2w-z_2$, then minus residue at $w = -1-2\epsilon$, then residue at $z_4 = 0$, and change the nature of the first pole of $\Gamma(z_2)$ (choose

a contour from the opposite side, i.e. the pole z_2 will be now to the right of the contour), then expand in ϵ . Denote this integral over z_1, z_2 and z_5 by B_2 .

3. Take minus residue at $z_3 = -2 - 2\epsilon - 2w - z_2$, then minus residue at $w = -1 - 2\epsilon$, then change the nature of the first pole of $\Gamma(z_4)$, then take a residue at $z_2 = -z_4$, then take a residue at $z_4 = -\epsilon$ and expand in ϵ . This resulting integral over z_1 and z_5 is denoted by B_3 .
4. Take minus residue at $z_3 = -2 - 2\epsilon - 2w - z_2$, then minus residue at $w = -1 - 2\epsilon$, then change the nature of the first pole of $\Gamma(z_4)$, then take a residue at $z_2 = -z_4$, then change the nature of the first pole of $\Gamma(2(\epsilon + z_4))$ and expand in ϵ . The resulting integral over z_1, z_4 and z_5 is denoted by B_4 .
5. Take minus residue at $z_3 = -2 - 2\epsilon - 2w - z_2$, then minus residue at $w = -1 - 2\epsilon$, then change the nature of the first pole of $\Gamma(z_4)$, then change the nature of the first pole of $\Gamma(z_2 + z_4)$ and expand in ϵ . The resulting integral over z_1, z_2, z_4 and z_5 is denoted by B_5 .
6. Take minus residue at $z_3 = -2 - 2\epsilon - 2w - z_2$, then change the nature of the first pole of $\Gamma(-2(1 + 2\epsilon + w))$, then take minus residue at $z_4 = 1 + w$, then minus residue at $z_2 = -1 - 2\epsilon - w$ and expand in ϵ . The resulting integral over w, z_1 and z_5 is denoted by B_6 .
7. Change the nature of the first pole of $\Gamma(-2 - 2\epsilon - 2w - z_2 - z_3)$, then take minus residue at $z_4 = -z_2 - z_3$, then a residue at $z_3 = 2\epsilon - z_2$, then take a residue at $z_2 = 2\epsilon$ and expand in ϵ . The resulting integral over w, z_1 and z_5 is denoted by B_7 .

One can see that all other contributions vanish at $\epsilon = 0$. By a suitable change of variables, one can observe that $B_7 = B_6$. In fact, the dependence of the first five contributions on the Mandelstam variable t is trivial: they are just proportional to $1/t$.

The two-dimensional integrals B_1 and B_3 are products of one-dimensional integrals which are taken by closing contour to the left and summing up resulting series with the help of formulae related to those of Appendix B.2 of [17]. The three-dimensional integral B_2 is evaluated by closing the integration contours over z_1 and z_5 to the left, summing up resulting series and applying a similar procedure to a final integral in z_2 . The corresponding result is naturally expressed through polylogarithms, up to $\text{Li}_3(x)$ depending on s and m^2 in terms of the variable

$$v = \left[\frac{\sqrt{1 - s/(4m^2)} + \sqrt{-s}/(2m)}{\sqrt{1 - s/(4m^2)} - \sqrt{-s}/(2m)} \right]^2.$$

The form of this result provides a hint about possible functional dependence of results for the three-(four-) dimensional integrals B_4 (B_5), and a heuristic procedure

which was explicitly formulated in [18] turns out to be successfully applicable here. First, all the contributions, in particular B_4 and B_5 , are analytic functions of s in a vicinity of the origin. One can observe that any given term of the Taylor expansion can be straightforwardly evaluated because the corresponding integrals over z_2 and z_4 are recursively taken. It is, therefore, possible to evaluate enough first terms (say, 30) of this Taylor expansion. Then one takes into account the type of the functional dependence mentioned above, turns to a new Taylor series in terms of the variable $v - 1$ and assumes that the n -th term of this Taylor series is a linear combination, with unknown coefficients, of the following quantities:

$$\frac{1}{n^4}, \frac{S_1(n)}{n^3}, \frac{S_2(n)}{n^2}, \frac{S_1^2(n)}{n^2}, \frac{S_3(n)}{n}, \dots,$$

where $S_k(n) = \sum_{j=1}^n j^{-k}$. (Here some terms of the fourth order are listed. See [18] for more details.) Using information about the first terms of the Taylor series one solves a system of linear equations, finds those unknown coefficients and checks this solution with the help of the next Taylor coefficients.

This experimental mathematics has turned out to be quite successful for the evaluation of B_4 and B_5 . Finally, the contribution B_6 is a product of a one-dimensional integral in z_1 , which is easily evaluated, and a two-dimensional integral in w and z_5 which involves a non-trivial dependence on t and is evaluated by closing the integration contours in z_5 to the left, summing up a resulting series in terms of Gauss hypergeometric function for which one can apply a parametric representation. After that the internal integral over w is taken by the same procedure and, finally, one takes the parametric integral.

The final result takes the following form:

$$B^{(0)}(s, t, m^2; \epsilon) = -\frac{(i\pi^{d/2}e^{-\gamma_E\epsilon})^2 x^2}{s^2(-t)^{1+2\epsilon}} \left[\frac{b_2(x)}{\epsilon^2} + \frac{b_1(x)}{\epsilon} + b_{01}(x) + b_{02}(x, y) + O(\epsilon) \right], \quad (5)$$

where $x = 1/\sqrt{1 - 4m^2/s}$, $y = 1/\sqrt{1 - 4m^2/t}$, and

$$b_2(x) = 2(m_x - p_x)^2, \quad (6)$$

$$\begin{aligned} b_1(x) = & -8 \left[\text{Li}_3\left(\frac{1-x}{2}\right) + \text{Li}_3\left(\frac{1+x}{2}\right) + \text{Li}_3\left(\frac{-2x}{1-x}\right) + \text{Li}_3\left(\frac{2x}{1+x}\right) \right] \\ & + 4(m_x - p_x) \left[\text{Li}_2\left(\frac{1-x}{2}\right) - \text{Li}_2\left(\frac{-2x}{1-x}\right) \right] - (4/3)m_x^3 + 4m_x^2 p_x \\ & - 6m_x p_x^2 + (2/3)p_x^3 + 4l_2(m_x p_x + p_x^2) - 2l_2^2(m_x + 3p_x) \\ & - (\pi^2/3)(4l_2 - m_x - 3p_x) + (8/3)l_2^3 + 14\zeta_3, \end{aligned} \quad (7)$$

$$\begin{aligned} b_{01}(x) = & -8(m_x - p_x) \left[\text{Li}_3(x) - \text{Li}_3(-x) - \text{Li}_3\left(\frac{1+x}{2}\right) + \text{Li}_3\left(\frac{1-x}{2}\right) \right] \\ & - \text{Li}_3\left(\frac{2x}{1+x}\right) + \text{Li}_3\left(\frac{-2x}{1-x}\right) + 4 \left[\text{Li}_2(x)^2 + \text{Li}_2(-x)^2 + 4\text{Li}_2\left(\frac{1-x}{2}\right)^2 \right] \end{aligned}$$

$$\begin{aligned}
& -8\text{Li}_2(x)\text{Li}_2(-x) + 16\text{Li}_2\left(\frac{1-x}{2}\right)(\text{Li}_2(x) - \text{Li}_2(-x)) \\
& -(4/3)[\pi^2 - 6l_2^2 + 3m_x^2 + 6m_x(2l_2 - 2l_x - p_x) + 12l_xp_x - 3p_x^2](\text{Li}_2(x) - \text{Li}_2(-x)) \\
& -(8/3)[\pi^2 - 6l_2^2 + 6l_xp_x - 6m_x(l_x + p_x - 2l_2)]\text{Li}_2\left(\frac{1-x}{2}\right) \\
& + 8(m_x - p_x)\left[(p_x - m_x + 2l_2)\text{Li}_2\left(\frac{2x}{1+x}\right) + 2(l_x - m_x + l_2)\text{Li}_2\left(\frac{-2x}{1-x}\right)\right] \\
& - 8(m_x - p_x)(2l_x - p_x - 5m_x + 4l_2)(-m_xp_x + l_2(m_x + p_x) - l_2^2 + \pi^2/6) \\
& - (20/3)m_x^4 + (164/3)m_x^3p_x - 40m_x^2p_x^2 - (4/3)m_xp_x^3 - (8/3)p_x^4 \\
& + 8m_xl_x(m_x^2 - 3m_xp_x + 2p_x^2) \\
& - 4l_2(7m_x^3 + 21m_x^2p_x - 4m_xl_xp_x - 23m_xp_x^2 + 4l_xp_x^2 - p_x^3) \\
& - \pi^2((17/3)m_x^2 - (4/3)m_xl_x - 2m_xp_x + (4/3)l_xp_x - (7/3)p_x^2) \\
& + l_2^2(84m_x^2 - 8m_xl_x - 16m_xp_x + 8l_xp_x - 44p_x^2) \\
& - (8/3)l_2(6l_2^2 - \pi^2)(3m_x - 2p_x) - (4/3)\pi^2l_2^2 + 4l_2^4 + \pi^4/9. \tag{8}
\end{aligned}$$

The last piece of the finite part comes from B_6 and B_7 :

$$\begin{aligned}
b_{02}(x, y) = & 2(p_x - m_x) \left\{ 4 \left[\text{Li}_3\left(\frac{1-x}{2}\right) - \text{Li}_3\left(\frac{1+x}{2}\right) + \text{Li}_3\left(\frac{(1-x)y}{1-xy}\right) \right. \right. \\
& \left. \left. - \text{Li}_3\left(\frac{-(1+x)y}{1-xy}\right) + \text{Li}_3\left(\frac{-(1-x)y}{1+xy}\right) - \text{Li}_3\left(\frac{(1+x)y}{1+xy}\right) \right] \right. \\
& + 2 \left[\text{Li}_3\left(\frac{(1+x)(1-y)}{2(1-xy)}\right) - \text{Li}_3\left(\frac{(1-x)(1+y)}{2(1-xy)}\right) \right. \\
& \left. \left. - \text{Li}_3\left(\frac{(1-x)(1-y)}{2(1+xy)}\right) + \text{Li}_3\left(\frac{(1+x)(1+y)}{2(1+xy)}\right) \right] \right. \\
& + 2(m_y + p_y - m_{xy} - p_{xy}) \left[2\text{Li}_2(x) - 2\text{Li}_2(-x) + \text{Li}_2\left(\frac{-2x}{1-x}\right) - \text{Li}_2\left(\frac{2x}{1+x}\right) \right] \\
& + 4(m_{xy} - p_{xy})(\text{Li}_2(-y) - \text{Li}_2(y)) - 4(m_x + p_x - 2l_2)\text{Li}_2\left(\frac{1-x}{2}\right) \\
& - 4(m_{xy} - p_{xy})\text{Li}_2\left(\frac{1-y}{2}\right) - 4(m_x + l_y - m_{xy})\text{Li}_2\left(\frac{(1-x)y}{1-xy}\right) \\
& + 4(p_x + l_y - m_{xy})\text{Li}_2\left(\frac{-(1+x)y}{1-xy}\right) - 4(m_x + l_y - p_{xy})\text{Li}_2\left(\frac{-(1-x)y}{1+xy}\right) \\
& + 4(p_x + l_y - p_{xy})\text{Li}_2\left(\frac{(1+x)y}{1+xy}\right) \\
& + 2(m_x + p_x + m_y + p_y - 2m_{xy} - 2l_2)\text{Li}_2\left(\frac{(1-x)(1+y)}{2(1-xy)}\right) \\
& + 2(m_x + p_x + m_y + p_y - 2p_{xy} - 2l_2)\text{Li}_2\left(\frac{(1-x)(1-y)}{2(1+xy)}\right)
\end{aligned}$$

$$\begin{aligned}
& +2p_x^2(m_y + p_y - m_{xy} - p_{xy}) + 2p_x(2(m_y l_y + m_y p_y + l_y p_y) \\
& + m_{xy}(-m_y - 2l_y - 3p_y + 3m_{xy}) + p_{xy}(-3m_y - 2l_y - p_y + 3p_{xy})) \\
& + 2m_x(2p_x + m_y - 2l_y + p_y)(m_y + p_y - m_{xy} - p_{xy}) - p_y^2(m_{xy} + p_{xy}) \\
& + 2p_y(2m_{xy}^2 + p_{xy}^2) + m_y^2(2p_y - m_{xy} - p_{xy}) \\
& + 2m_y(p_y^2 + m_{xy}^2 + 2p_{xy}^2 - p_y(3m_{xy} + p_{xy})) - 2(m_{xy}^3 + p_{xy}^3) \\
& + 2l_2((4m_y + 4p_y - 3m_{xy})m_{xy} + (2m_y + 2p_y - 3p_{xy})p_{xy} \\
& - 2(p_x + 2m_x)(m_y + p_y - m_{xy} - p_{xy}) - m_y^2 - 4m_y p_y - p_y^2) \\
& + 2l_2^2(3(m_y + p_y) - 2(2m_{xy} + p_{xy})) \\
& - (\pi^2/3)(m_y + p_y - 8m_{xy} + 6p_{xy}) \} . \tag{9}
\end{aligned}$$

Here $\text{Li}_a(z)$ is a polylogarithm [19]. The following abbreviations are also used: $\zeta_3 = \zeta(3)$, $l_z = \ln z$ for $z = x, y, 2$, $p_z = \ln(1 + z)$ and $m_z = \ln(1 - z)$ for $z = x, y, xy$.

This result is presented in such a way that it is manifestly real at small negative values of s and t . From this Euclidean domain, it can be easily analytically continued to any other domain.

3. The result (5)–(9) is in agreement with the leading power behaviour in the (Sudakov) limit of the fixed-angle scattering, $m^2 \ll |s|, |t|$. This asymptotics is obtained by use of the strategy of expansion by regions [13, 14, 15]. The structure of regions is very rich. The following family of seventeen regions participates here:

$$\begin{aligned}
& (h-h), (1c-h), \dots, (4c-h), (1c-1c), \dots, (4c-4c), \\
& (1c-3c), (2c-4c), (1c-4c), (2c-3c), \\
& (1uc-2c), (2uc-1c), (3uc-4c), (4uc-3c).
\end{aligned}$$

Here h denotes hard, c – collinear and uc – ultracollinear regions for the two loop momenta. (See [14] and Chapter 8 of [15] for definitions of these regions.) In particular, the $(h-h)$ contribution is nothing but the massless on-shell double box [2]. The contributions $(1c-1c)$, $(3c-3c)$, $(1c-3c)$ as well as the symmetrical contributions $(2c-2c)$, $(4c-4c)$ and $(2c-4c)$ are not individually regularized by dimensional regularization. The poles in the auxiliary analytic regularization turn out to be of the second order and are cancelled in the sum. After adding the rest of the contributions, the poles of the third and fourth order in ϵ are cancelled. Following this procedure, we obtain

$$\begin{aligned}
B^{(0)}(s, t, m^2; \epsilon) = & -\frac{(i\pi^{d/2}e^{-\gamma_E\epsilon})^2}{s^2(-t)^{1+2\epsilon}} \left\{ 2\frac{L^2}{\epsilon^2} - \left[(2/3)L^3 + (\pi^2/3)L + 2\zeta_3 \right] \frac{1}{\epsilon} \right. \\
& - (2/3)L^4 + 2\ln(t/s)L^3 - 2(\ln^2(t/s) + 4\pi^2/3)L^2 \\
& + \left[4\text{Li}_3(-t/s) - 4\ln(t/s)\text{Li}_2(-t/s) + (2/3)\ln^3(t/s) - 2\ln(1+t/s)\ln^2(t/s) \right. \\
& \left. \left. + (8\pi^2/3)\ln(t/s) - 2\pi^2\ln(1+t/s) + 10\zeta_3 \right] L + \pi^4/36 \right\} + O(m^2 L^3, \epsilon), \tag{10}
\end{aligned}$$

where $L = \ln(-m^2/s)$. This asymptotic behaviour is reproduced when one starts from result (5)–(9).

The analytical result presented above agrees also with results based on numerical integration in the space of alpha parameters [20] (where the 1% accuracy for the $1/\epsilon$ and ϵ^0 parts is guaranteed).

Let us stress that, in the present case with a non-zero mass, there are no collinear divergences and the poles in ϵ are only up to the second order, so that the resolution of singularities in ϵ in the MB integrals is relatively simple. Therefore, it looks promising to use the technique presented, starting from (3), for the evaluation of any given master integral. The construction of a recursive algorithm that would express any given planar double box through some family of master integrals is one of the next problems, as well as similar problems for the evaluation of massive on-shell non-planar double-box Feynman integrals. Another possible scenario in the situation, where the mass is small but still not negligible, is to evaluate the leading power (and all logarithms) asymptotics when $m \rightarrow 0$. Anyway, the (h-h) contribution to any such asymptotic behaviour is obtained by the algorithms of [2, 3, 4, 5, 6].

It is interesting to note that, in the above result, there are no so-called two-dimensional harmonic polylogarithms [21] which have turned out to be adequate functions to express results for the double boxes with one leg off-shell [10]. It is also an open question whether this phenomenon takes place for general massive on-shell double boxes.

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